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Extreme Varieties, Concave Functions, and The Fixed Charge Problem

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EXTREME VARIETIES, CONCAVE FUNCTIONS, AND THE FIXED CHARGE PROBLEM

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1. Preliminaries and Notation

This paper deals with various structural properties of convex sets and their application to the study of concave mappings. The investigation was prompted by a programming problem (the fixed charge problem) which involves the minimization of a concave function on a convex polyhedron. In the fixed charge problem it is not difficult to show (by an argument involving the form of the function and the polyhedral character of the domain) that the desired infimum is actually attained [8]. On the other hand it is not generally true that a concave function on a convex set attains its infimum, and it is natural to seek conditions under which this is so. The search for such conditions leads to the study of representations of convex sets similar to the Krein-Milman theorem.

In Section 2 we introduce the notion of extreme variety, a generalization of extreme points, and use it to obtain a decomposition of an arbitrary convex set into the vector sum of a linear variety, a convex cone, and the convex hull of the extreme points of a related convex set. In Section 3 we exploit this decomposition to derive simple sufficient conditions for the attainment of the infimum of a concave function. In particular we show that if the domain of the function is a convex polyhedron and the infimum is finite, it is necessarily attained. In Section 4 we formulate the fixed charge problem and, applying the previous results, demonstrate the existence of a solution.

All sets that we consider are subsets of a real finite dimensional Euclidean space R. Points are denoted by lower case letters and sets by capitals. For the open interval having endpoints x and y we use the customary notation (x,y). The same notation is used for inner product, the meaning of the symbol always being clear from the context. The signs - and $\sim$ denote, respectively, vector difference and set-theoretic difference. For the vector sum of sets $A_1, A_2, \ldots, A_n$ we use either $A_1 + A_2 + \ldots + A_n$ or $\sum_{j=1}^{n} A_j$ and for set-theoretic intersection either $A_1 \cap A_2 \cap \ldots \cap A_n$ or $\bigcap_{j=1}^{n} A_j$. The orthogonal complement of a linear space L is denoted by $L^{\perp}$.

By the convex hull of a set of points G we mean the minimal convex set comtaining G. A linear variety, V, is a translate of a linear space. A hyperplane relative to V is a maximal proper linear variety contained in V, i.e., if V is of dimension d, a hyperplane is any d-1 dimensional linear subvariety. A section of a convex set K is the intersection of K with a linear variety.

A non-empty subset C of R is called a cone if $x \in C$ implies $\lambda x \in C$ for every non-negative real number λ. A cone is convex if $x \in C$ and $y \in C$ implies $(x+y) \in C$.

With any convex set K there is associated a smallest linear variety R(K) of R which contains K. The dimension d(K) of the variety R(K) is called the dimension of K. In the theorems which follow R(K) plays a more important role than R. For brevity we shall refer to a hyperplane relative to R(K) as a relative hyper-

plane and to points which are boundary points of K relative to R(K) as relative boundary points.

2. Structure of Convex Sets

The notion of extreme point has emerged within recent years as a central one in studies of convexity, one of the basic results concerning such points being the Krein-Milman theorem, which asserts that a closed, bounded convex set K is the convex hull of its extreme points. If the boundedness assumption is relinquished, K need not have any extreme points, and the Krein-Milman result obviously does not apply. In this case, however, Klee [9] has shown that K is generated by its extreme points and extremal rays, thus generalizing the Krein-Milman theorem.

Our goal in this part of the paper is to obtain a generalization of a different form, one that is adapted to the application referred to in the introductory section. We shall show that every closed convex set K has a section M which has at least one extreme point and is such that K can be represented as a vector sum in which M is one of the components. In turn M will be seen to be the vector sum of the convex hull H of its extreme points and an associated cone. The advantage of the resulting representation of K lies in the fact that the search for "small" values of a concave function on K can be restricted to the set H.

We begin by introducing the concept of an "extreme variety": A linear variety $V \subset K$ is called extreme if $K \sim V$ is convex, or equivalently, if $p \in V$ and $p \in (x,y) \subset K$ imply $(x,y) \subset V$. Clearly

an extreme point is an extreme variety of dimension zero. Although a closed convex set need not contain an extreme point, it must always contain an extreme variety, and all its extreme varieties are parallel spaces of the same dimension, as Theorem 2.3 shows. To smooth the way for the proof of this result and its consequences we state as lemmas the following obvious facts:

2.1. LEMMA. If W and W^* are linear varieties in K, and W is a proper subset of W^*, then W is not an extreme variety.

Proof: Clearly each point in W is a convex combination of two points in $W^* \sim W$. Hence $K \sim W$ is not convex.

2.2. LEMMA. If K contains a translate of a cone C, then

$$K = K + C \ .$$

In the special case when C is a linear space, K may be written in the form

$$K = K + C = K + c \ ,$$

where $c \in C$ is arbitrary.

Proof: Clearly $K \subset K + C$. The converse is a consequence of the closure of K. For, by hypothesis, there is a point p such that

$$p + C \subset K \ .$$

Hence, if $q \in K$, $r \in C$, and $\mu \geq 1$ are chosen arbitrarily, we have

$$(1-\frac{1}{\mu})q+\frac{1}{\mu}(p+\mu r)\in K .$$

Letting $\mu \longrightarrow \infty$, it follows that

$$q+r\in K .$$

Thus

$$K = K+C .$$

If C is a linear space, and $c\in C$ is arbitrary, we have for each point $k\in K$,

$$k=[(k-c)+c]\in[(K+C)+c] .$$

Since $K+C=K$, it follows that

$$K\subset K+c .$$

Obviously

$$K+c\subset K+C=K ,$$

and hence

$$K=K+c .$$

2.3. THEOREM. Every closed convex set K contains an extreme variety. Moreover, all extreme varieties of K are translates of each other, each having the dimension of the linear variety of maximum dimension in K.

Proof: Let $W=p+L$ and $W^*=p^*+L^*$ be linear varieties in K. If L is not contained in L^*, then W^*+L properly contains W^* and by Lemma 2.2 is contained in K. It follows from Lemma 2.1

that W^* is not an extreme variety. Thus a necessary condition for W^* to be an extreme variety is that

$$L \subset L^* ,$$

which establishes the assertion that an extreme variety is of maximal dimension. It proves also that if W and W^* are two extreme varieties, then

$$L = L^* ,$$

and consequently

$$W^* = (p^* - p) + W .$$

To complete the proof we need only demonstrate the existence of an extreme variety in K. We do this by induction on the dimension d of K. Clearly the assertion is true if $d = 0$. Suppose it has been proved for $d < n$, and suppose K is n-dimensional. If K has no relative boundary points, then $K = R(K)$, and K is an extreme variety of itself.

Assume, therefore, that K has a relative boundary point p. Let S be a relative hyperplane which supports K at p. By the inductive hypothesis, $S \cap K$ admits an extreme variety W. Let q be any point such that

$$q \in W , \tag{1}$$

and, for some interval $(x,y) \subset K$,

$$q \in (x,y) . \tag{2}$$

Since $q \in S$, and S supports K, we must have

$$(3) \qquad (x,y) \subset S \cap K .$$

It follows that

$$(4) \qquad (x,y) \subset W ,$$

since W is an extreme variety of $S \cap K$. Finally, since (1) and (2) imply (4), we conclude that W is an extreme variety of K.

A very useful special case of Theorem 2.3 is the following:

2.4. COROLLARY. A closed convex set K contains an extreme point if and only if it contains no line.

Proof: The corollary follows from the observation that extreme points are extreme varieties of dimension zero, and these are the only such extreme varieties.

As the following counter-example shows, Theorem 2.3 does not hold in an infinite-dimensional linear topological space without further assumptions about the structure of the space. Let $\mathcal{C}_0$ be the space of real-valued, continuous functions $\theta(t)$ defined on the interval $0 \leq t \leq 1$ and vanishing at $t = 0$. With the norm defined by

$$\|\theta\| = \sup_{0 \leq t \leq 1} |\theta(t)| ,$$

consider the unit solid sphere K. Obviously K is closed and contains no linear variety of positive dimension. It is easy to see that it contains no extreme points. For if θ is an arbitrary

point in $\mathcal{C}_0$, and α satisfies the inequality $0 < \alpha < 1$, there exists a real number τ, $0 < \tau < 1$, such that

$$|\theta(t)| < 1 - \alpha$$

for all t in the interval

$$0 \leq t \leq \tau .$$

Then putting

$$\underline{\theta}(t) = \begin{cases} \theta(\tau) - \alpha t & , \quad 0 \leq t \leq \frac{\tau}{2} \\ \theta(\tau) - \alpha(\tau - t) & , \quad \frac{\tau}{2} \leq t \leq \tau \\ \theta(\tau) & , \quad \tau \leq t \leq 1 \end{cases}$$

and

$$\overline{\theta}(t) = \begin{cases} \theta(\tau) + \alpha t & , \quad 0 \leq t \leq \frac{\tau}{2} \\ \theta(\tau) + \alpha(\tau - t) & , \quad \frac{\tau}{2} \leq t \leq \tau \\ \theta(\tau) & , \quad \tau \leq t \leq 1 , \end{cases}$$

we have $\underline{\theta}(\tau) \in K$, $\overline{\theta}(\tau) \in K$, and

$$\theta = \frac{1}{2}\underline{\theta} + \frac{1}{2}\overline{\theta} .$$

Hence θ is not an extreme point.

We turn next to a decomposition theorem for an arbitrary closed convex set K, which shows that each such set is the vector sum of a section having at least one extreme point and a linear space. By imposing the additional requirement that the two components of the sum be orthogonal, the decomposition become unique.

2.5. THEOREM. Let K be a non-empty closed convex set. There exists a unique decomposition of K having the form

$$(5) \qquad K = L + M ,$$

where L is a linear space, M is a closed convex subset of $L^{\perp}$, and M contains at least one extreme point.

Remark. It is clear from (5) that M is a section of K, namely, $M = K \cap L^{\perp}$. We refer to L as the linear space determined by K and to M as the $L^{\perp}$-section of K.

Proof: By Theorem 2.3 K contains an extreme variety, say $p + L$. Decomposing p into orthogonal components

$$p = \ell_p + m_p$$

where $\ell_p \in L$ and $m_p \in L^{\perp}$, we can express the extreme variety in the form

$$p + L = m_p + L .$$

Set

$$M = L^{\perp} \cap K .$$

It follows from Lemma 2.2 that

$$L + M \subset K .$$

Conversely, expressing an arbitrary point $k \in K$ in the form

$$k = \ell_k + m_k ,$$

where $\ell_k \in L$ and $m_k \in L^{\perp}$, we see from Lemma 2.2 that

$$m_k = k - \ell_k \in K ,$$

and therefore

$$K \subset L + M .$$

Thus,

$$K = L + M .$$

We observe next that m_p is an extreme point of M. For, if there is an interval (x,y) such that

$$m_p \in (x,y) \subset M \subset K ,$$

then

$$(x,y) \subset m_p + L , \tag{6}$$

since $m_p + L$ is an extreme variety of K. But (6), together with the fact that

$$(x,y) \subset L^{\perp} ,$$

implies

$$(x,y) = m_p .$$

Hence, m_p is an extreme point of M.

To prove the uniqueness of the representation (5) suppose that there are two such decompositions:

$$K = L + M = \hat{L} + \hat{M} . \tag{7}$$

Let m be any extreme point of M. We shall show that the linear variety

$$V = L + m$$

is an extreme variety of K. Assume that $q \in V$ and $q \in (x,y) \subset K$. Then

$$q = \ell + m = \frac{1}{2}(r+s) \ ,$$

where $\ell \in L$, $r \in (x,y)$, and $s \in (x,y)$. By (7) we can express r and s, respectively, in the forms

$$r = \ell_1 + m_1$$

and

$$s = \ell_2 + m_2 \ ,$$

where $\ell_j \in L$ and $m_j \in M$, $j = 1,2$. It follows that

$$m - \frac{1}{2}(m_1 + m_2) = \frac{1}{2}(\ell_1 + \ell_2) - \ell \ . \tag{8}$$

The vectors on the right-hand side of (8) are orthogonal to those on the left. Hence,

$$m = \frac{1}{2}(m_1 + m_2) \ . \tag{9}$$

Since m is an extreme point of M, (9) implies

$$m_1 = m_2 = m \ .$$

Thus

$$r = \ell_1 + m \in V \ ,$$

and

$$s = \ell_2 + m \in V \ .$$

It follows that $(x,y) \subset V$, and V is an extreme variety. The same argument shows that an appropriate translate of $\hat{L}$ is an extreme variety, and since all extreme varieties are parallel to each other, we must have

$$L = \hat{L} .$$

Finally,

$$M = K \cap L^{\perp} = K \cap \hat{L}^{\perp} = \hat{M} ,$$

which completes the proof of the theorem.

We are now in a position to obtain a more complete decomposition of a closed convex set K, which generalizes (in finite dimensional space) the Krein-Milman theorem. To provide a succinct statement of the result we introduce the concept of a cone determined by a closed convex set: Let $p \in K$ be arbitrary, and let $C_K(p)$ be the set of all points x such that

$$p + \lambda x \in K$$

for all $\lambda \geq 0$. Although p ostensibly enters the definition of $C_K(p)$, the set is in fact independent of p. For clearly $C_K(p)$ is a cone such that $p + C_K(p) \subset K$. Hence, by Lemma 2.2, if $x \in C_K(p)$ and $q \in K$ are arbitrarily chosen, we have

$$q + \lambda x \in K$$

for all $\lambda \geq 0$. It follows that $x \in C_K(q)$, and therefore

$$C_K(p) \subset C_K(q) .$$

Since p and q play symmetric roles,

$$C_K(p) = C_K(q) .$$

In the sequel we denote this cone by C_K and refer to it as the cone determined by K. It is easy to see that C_K is closed and convex.

According to Theorem 2.5 K determines a linear space L such that the $L^{\perp}$-section of K has an extreme point. Denoting this section by M, we note that since M contains an extreme point, C_M is a pointed cone. We refer to C_M as the pointed cone determined by K and observe that

$$C_M = C_K \cap L^{\perp} .$$

Our main result concerning the decomposition of K can now be expressed as follows:

2.6. THEOREM[1]. A closed convex set K is the vector sum of the linear space L it determines, the pointed cone C_M it determines, and the convex hull H_M of the extreme points of M, i.e.,

$$K = L + C_M + H_M .$$

Proof: By Theorem 2.5 it suffices to show that

$$M = H_M + C_M .$$

From the definition of C_M we see that

$$H_M + C_M \subset M .$$

We shall prove that

$$M \subset H_M + C_M \tag{10}$$

by induction on the dimension d(M) of M. Obviously (10) is true if $d(M) = 0$. Assume it has been proved for $d(M) < n$ and suppose $d(M) = n$. Let $p \in M$ be arbitrary, and let L be any line in R(M),

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[1] In the case of a convex polyhedron containing at least one extreme point, the decomposition described in this theorem was observed to hold by A. Tucker [6,7], but his methods do not seem to apply to the general case.

the smallest linear variety containing M, which passes through p. Since M is closed and (by Corollary 2.4) contains no line, $L \cap M$ is either a closed half-line or a closed bounded line segment.

If $L \cap M$ is a half-line, it has an endpoint t which we can express in the form

$$t = p + \lambda^* u ,$$

where $u \in R(M)$, $\lambda^* \leq 0$, and for all $\lambda \geq \lambda^*$, $p + \lambda u \in L \cap M$. Clearly t is a relative boundary point of M. Hence there is a relative hyperplane S which contains t and supports M. Applying the inductive hypothesis to $S \cap M$ we conclude that

$$t \in H_{S \cap M} + C_{S \cap M} .$$

It is clear from the definition of C that

$$C_{S \cap M} \subset C_M ,$$

and since S supports M,

$$H_{S \cap M} \subset H_M .$$

Hence

$$t \in H_M + C_M .$$

Observing that

$$-\lambda^* u \in C_M ,$$

it follows that

$$p = t - \lambda^* u \in (H_M + C_M) + C_M = H_M + C_M .$$

If $L \cap M$ is a bounded line segment, it has endpoints t and t' (which coincide if $L \cap M$ consists of a single point), which, by the argument leading to (11), both belong to $H_M + C_M$. Since $H_M + C_M$ is convex, and p is a convex combination of t and t', we have

$$p \in H_M + C_M \ .$$

This concludes the proof.

3. The Infimum of a Concave Function

In this section we apply the preceding results to the problem of determining conditions under which a concave function ψ on a closed convex set K attains its infimum. Assume that ψ is bounded below. Without further assumptions about ψ or K it is evident that ψ need not assume its infimum. Consider, for example, the set K in 2-space consisting of pairs (x_1,x_2) such that $x_1 > 0$, $x_2 > 0$, and $x_1x_2 \geq 1$. Let $\psi(x_1,x_2) = x_2$. Then,

$$\inf_{(x_1,x_2)\in K} \psi(x_1,x_2) = 0 \ ,$$

which is not attained. As the following theorem shows, the attainment or non-attainment of the infimum is determined by the behavior of ψ on the extreme points of $L^{\perp}$-section of K, where $L^{\perp}$ is the orthogonal complement of the linear space determined by K.

3.1. THEOREM. Let the concave function ψ on the non-empty closed convex set K be bounded below. ψ assumes its infimum σ if and only if the restriction of ψ to the set E_M of extreme points

If L ∩ H is a bounded line segment, it has endpoints t and t' (which coincide if L ∩ H consists of a single point), which, by the argument leading to (11), both belong to $C_1 = C_2$. Since $C_1 = C_2$ is convex, and p is a convex combination of t and t', we have

$$p \in C_1 = C_2 .$$

This concludes the proof.

7. The Maximum of a Concave Function

In this section we apply the preceding results to the problem of determining conditions under which a concave function ϕ on a closed convex set K attains its maximum. Assume that ϕ is bounded above. Without further assumptions about ϕ or K it is evident that ϕ need not attain its supremum. Consider, for example, the set K in 2-space consisting of points (x_1,x_2) such that $x_1 > 0$, $x_2 > 0$, and $x_1 x_2 \geq 1$. Let $\phi(x_1,x_2) = -x_1$. Then,

$$\sup_{(x_1,x_2)\in K} \phi(x_1,x_2) = 0 ,$$

which is not attained. As the following theorem shows, the attainment or non-attainment of the maximum is determined by the behavior of ϕ on the extreme points of $L^{\perp}$-section of K, where $L^{\perp}$ is the orthogonal complement of the linear space determined by K.

7.1. THEOREM. Let the concave function ϕ on the non-empty closed convex set K be bounded above. ϕ attains its maximum, if and only if the restriction of ϕ to the set E_K of extreme points of $M = K \cap L^{\perp}$ assumes its infimum. When it is assumed, σ is assumed on E_M.

Proof: It suffices to show that to each point $p \in K$ there corresponds an extreme point $e \in E_M$ such that

$$\psi(p) \geq \psi(e) \; . \tag{12}$$

To this end we express p in the form

$$p = \ell + c + h \; ,$$

where (using the notation of Theorem 2.6) $\ell \in L$, $c \in C_M$, and $h \in H_M$. Since the function

$$\psi^*(\ell) = \psi(\ell + c + h)$$

is concave, bounded from below, and has the linear space L as its domain, ψ^* is a constant. Thus,

$$\psi^*(\ell) = \psi^*(0) \; ,$$

that is,

$$\psi(p) = \psi(c + h) \; .$$

We shall show that

$$\psi(p) \geq \psi(h) \; .$$

Let $\lambda > 1$. Then, since h and $h + \lambda c$ both belong to M, and $c + h = \frac{\lambda-1}{\lambda} h + \frac{1}{\lambda}(h + \lambda c)$, we have

$$\psi(p) = \psi(c + h) \geq \frac{\lambda-1}{\lambda} \psi(h) + \frac{1}{\lambda} \psi(h + \lambda c) \geq \frac{\lambda-1}{\lambda} \psi(h) + \frac{1}{\lambda} \sigma \; .$$

Letting $\lambda \longrightarrow \infty$, we conclude that

$$\psi(p) \geq \psi(h) .$$

Since $h \in H_M$, there exist extreme points $e_1, \ldots, e_r$ of M such that

$$h = \sum_{j=1}^{r} \lambda_j e_j ,$$

where $\lambda_j \geq 0$ and $\sum_{j=1}^{r} \lambda_j = 1$. Hence

$$\psi(p) \geq \psi(h) \geq \sum_{j=1}^{r} \lambda_j \, \psi(e_j) \geq \min_{1 \leq j \leq r} \psi(e_j) = \psi(e_{j_o}) ,$$

where $1 \leq j_o \leq r$. This establishes (12) and completes the proof.

3.2. COROLLARY. ψ assumes the infimum if M, the $L^{\perp}$-section of K, has a finite number of extreme points. In particular, ψ assumes its infimum on a closed convex polyhedron.

3.3. COROLLARY.[2] ψ assumes its infimum if H_M, the convex hull of the extreme points of the $L^{\perp}$-section of K, is compact and ψ is lower semi-continuous on H_M.

Proof: Let $\{\varepsilon_n\} \downarrow 0$ be a sequence of positive numbers and $\{x_n\}$ a sequence of points in H_M such that

$$\psi(x_n) < \sigma + \varepsilon_n .$$

---

[2] This corollary remains valid if the set H_M appearing in it is replaced either by E_M, the set of extreme points of M, or by K, the convex set itself. However, if E_M is compact, it follows that H_M is compact (but not conversely), and if K is compact, $K = H_M$. Hence 3.3 is the most general statement.

Without loss of generality we assume that $\{x_n\}$ converges, and we put

$$\lim_{n \to \infty} x_n = x_o ,$$

where

$$x_o \in H_M .$$

Let $\varepsilon > 0$ be arbitrary. Then for all sufficiently large values of n,

$$\psi(x_n) > \psi(x_o) - \varepsilon .$$

Thus

$$\sigma + \varepsilon_n > \psi(x_o) - \varepsilon .$$

Letting $n \longrightarrow \infty$ and $\varepsilon \longrightarrow 0$, it follows that

$$\sigma \geq \psi(x_o) .$$

Since $\sigma = \inf_{x \in K} \psi(x)$,

$$\psi(x_o) = \sigma .$$

4. The Fixed Charge Problem[3]

There is an important problem in the programming area to which the preceding results are relevant. It arises in any situation which involves the planning of a large number of interdependent activities, some (or all) of which have set-up time charges. Consider, for example, a factory in which n distinct operations are performed by each of m machines. Let h_{ij} and c_{ij},

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[3] The authors wish to acknowledge the participation of George B. Dantzig in the work described in this section.

$i = 1,\ldots,m$, $j = 1,\ldots,n$, denote, respectively, the time consumed and the cost incurred by the j^{th} machine in performing the i^{th} operation on a unit amount of goods. Suppose that there is incurred a fixed cost, say b_{ij}, in setting up the j^{th} machine to perform the i^{th} operation. (We ignore the time involved in this activity.) Let x_{ij} denote the quantity of goods on which the i^{th} operation is performed by the j^{th} machine, and b_i the total quantity of goods on which it is required to perform the i^{th} operation. Evidently the quantities x_{ij} must satisfy the conditions

$$\sum_{j=1}^{n} x_{ij} = b_i \,, \tag{13}$$

and

$$x_{ij} \geq 0 \,, \qquad \begin{array}{l} i = 1,\ldots,m \\ j = 1,\ldots,n \,. \end{array} \tag{14}$$

Moreover, if c_j denotes the total machine hours available for the j^{th} machine, we have the additional constraints

$$\sum_{i=1}^{m} h_{ij}\, x_{ij} \leq c_j \,, \qquad j = 1,\ldots,n \,. \tag{15}$$

The total cost of operation is given by the function

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij} + b_{ij}\, \delta(x_{ij}) \,, \tag{16}$$

where

$$\delta(u) = \begin{cases} 1 \,, & u > 0 \\ 0 \,, & u = 0 \,. \end{cases}$$

The problem is to minimize (16), the cost function, subject to the constraints (13), (14), and (15). The difficulty is due to the fact that (16) is a sum of functions each of which has a jump discontinuity at the origin.

More generally, let $q, p_1, \ldots, p_n$ be points in R_m, and let X denote the set of all points $x = (x_1, \ldots, x_n)$ in R_n $(n > m)$ which satisfy the relations

$$\sum_{j=1}^{n} x_j p_j = q \tag{17}$$

and

$$x_j \geq 0 \,, \qquad j = 1, \ldots, n \,. \tag{18}$$

Put

$$\psi(x) = \sum_{i=1}^{n} \alpha_i x_i + \beta_i \;\delta(x_i) \,, \tag{19}$$

where $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n$ are constants. We refer to the problem of minimizing ψ over the set X as the fixed charge problem and to the constant β_i as the fixed charge associated with the i^{th} activity.

In one very special case it is possible to reduce the fixed charge problem to that of linear programming, namely, when the fixed charges are all equal and non-negative.

4.1. THEOREM. Let

$$\psi = \sum_{i=1}^{n} \alpha_i x_i + \beta \sum_{i=1}^{n} \delta(x_i) \,, \qquad \beta \geq 0 \,,$$

and

$$\psi^* = \sum_{i=1}^{n} \alpha_i x_i \,.$$

Suppose that the point q appearing in (17) does not lie in any subspace of R_m spanned by fewer than m of the points $P_1,\ldots,P_n$ (non-degeneracy assumption). Then there exists an extreme point $\hat{x} \in X$ having exactly m positive components at which both ψ and ψ^* are minimized. Moreover, any point x with m positive components which minimizes ψ^* also minimizes ψ.[4]

Proof: Under the non-degeneracy assumption any vector x satisfying (17) must have at least m non-zero components. Since the sum, $\sum_{i=1}^{n} \delta(x_i)$, is equal to the number of non-zero components of x, we have

$$\text{(20)} \qquad \inf_{x \in X} \psi(x) \geq \inf_{x \in X} \psi^*(x) + \beta \inf_{x \in X} \sum_{i=1}^{n} \alpha(x_i) \geq \inf_{x \in X} \psi^*(x) + m\beta \ .$$

It is well-known from the theory of linear programming that under the assumption of non-degeneracy there is an extreme point $\hat{x} \in X$ having exactly m positive components at which ψ^* is a minimum. At such a point ψ assumes the value

$$\psi(\hat{x}) = \psi^*(\hat{x}) + m\beta \ .$$

Hence, from (20)

$$\psi(\hat{x}) = \psi^*(\hat{x}) + m\beta \geq \inf_{x \in X} \psi(x) \geq \inf_{x \in X} \psi^*(x) + m\beta = \psi^*(\hat{x}) + m\beta = \psi(\hat{x}) \ ,$$

which completes the proof.

---

[4] Since the simplex method leads to a point $\hat{x}$ with m positive components at which ψ^* is a minimum, this method can also be used to solve the fixed charge problem with a constant positive fixed charge.

In the general case, when the fixed charges β_i vary with i, there is no expedient computational algorithm, but as the following theorem asserts, the minimum is assumed at an extreme point.

4.2. THEOREM. If the function ψ defined in (19) is bounded below and $\beta_i \geq 0$, $i = 1,\ldots,n$, then ψ assumes its infimum σ. Moreover, σ is assumed at an extreme point.

Proof: Since ψ is concave, and the domain defined by (17) and (18) is a closed convex polyhedron, the theorem follows from Corollary 3.2.

In certain logistical problems [8] the cost function is a generalization of (19) having the form

$$(21) \qquad \psi(x) = \sum_{j=1}^{n} \alpha_j x_j + \sum_{r=1}^{p} \beta_r \, \delta\left(\sum_{j=(r-1)k+1}^{rk} x_j \right) ,$$

where n = kp. Clearly (21) reduces to (19) when p = n and k = 1. We refer to the problem of minimizing (21) under the constraints (17) and (18) as the generalized fixed charge problem. For this problem there is no analogue of Theorem 4.1, but once again Corollary 3.2 shows that the infimum is assumed at an extreme point.

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